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NON-LINEAR DYNAMICS OF COUPLED CHAINS OF PARTICLES†

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The equations of motion of two linear periodic chains of non-linearly interacting particles are considered in the long-wave approximation. The system of equations obtained is a model for describing wave processes in two-component media. Methods of group analysis (see, for example, [1]) are used to pick out the submodels that admit of the largest group of point mappings. Particular invariant solutions are presented resented for two submodels with obvious mechanical interpretations. It is shown that, if the potential of the non-linear interaction can be expressed as a harmonic function of the relative displacement of particles in the chains, and the acoustic velocities of non-interacting chains are different, the system is a special type of soliton filter; the allowed soliton velocities are determined. A few solutions describing the long-wave dynamics of the system are presented, assuming the presence of additional shear forces.

1. Consider two coupled linear periodic chains of particles (see Fig. 1), where the mass of any particle of the "upper" chain is m_1 and of the "lower" chain m_2. In the equilibrium configuration the distance between adjacent particles in each chain is a. The particles may move only along smooth tracks parallel to the X axis. The interaction between next neighbours in the chains is considered in the usual harmonic approximation, but the interaction constants may be different: β_1 and β_3. The function characterizing the interaction between the chains depends on the displacements of matching particles (P_n^1 and P_n^2). The precise form of the function is for the moment not essential; it will be an arbitrary element in the group classification problem to be considered below.

Let u_n be the displacement of particle P_n^1 and w_n that of particle P_n^2. The dynamics of the system is described by the equations

$$\begin{aligned} m_1\ddot{u}_n &= \beta_1(u_{n+1}-2u_n+u_{n-1})-\partial H(u_n,w_n)/\partial u_n \\ m_2\ddot{w}_n &= \beta_2(w_{n+1}-2w_n+w_{n-1})-\partial H(u_n,w_n)/\partial w_n \end{aligned} \tag{1.1}$$

(the dot stands for differentiation with respect to time); $H(u_n, w_n)$ is the energy of interaction of P_n^1 and P_n^2.

Changing to dimensionless variables

$$\tilde{t}=\frac{c_1}{a}t,\quad \tilde{x}=\frac{x}{a},\quad \tilde{u}=\frac{u}{a},\quad \tilde{w}=\left(\frac{m_2}{m_1}\right)^{1/2}\frac{w}{a},\quad \tilde{H}=\frac{H}{m_1c_1^2}$$

$$\left(c^2=\frac{c_2^2}{c_1^2}=\frac{\beta_2 m_1}{\beta_1 m_2},\quad c_i^2=\frac{\beta_i a^2}{m_i},\quad i=1,2\right)$$

introducing the force function $f(\tilde{u},\tilde{w})=-\tilde{H}(\tilde{u},\tilde{w})$ and taking the long-wave approximation, we deduce from (1.1) the following system of partial differential equations (the tilde is omitted)

$$u_{tt}-u_{xx}=f_u(u,w),\quad w_{tt}-c^2w_{xx}=f_w(u,w) \tag{1.2}$$

2. Table 1 lists the results of classifying equations (1.2) by groups of point transformations. If $f_{uw}(u, w) = 0$, system (1.2) splits into two independent Klein–Gordon equations, whose group classification has already been carried out by Lie (see, for example, [2]). The case $c^2 = 1$ has

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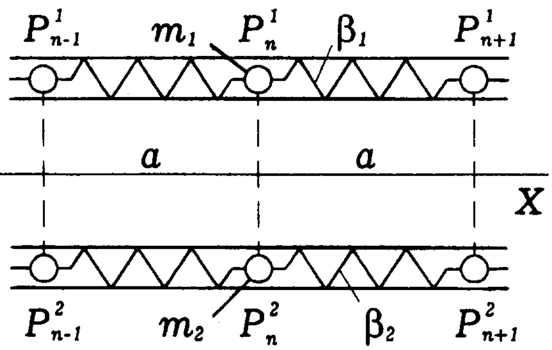

Fig. 1.

been investigated [3], the highest symmetries of the equations have been determined and cases of complete or partial integration have been described. These two cases will therefore be omitted in what follows.

The classification is carried out up to all continuous equivalence transformations obtained by the infinitesimal method [1] and the obvious discrete transformations

$$
\begin{aligned}
&\tilde{t} = a_1 t + a_2, \quad \tilde{x} = a_1 x + a_3, \quad \tilde{u} = a_4 u + a_5, \quad \tilde{w} = a_4 w + a_6, \quad \tilde{f} = a_4^2 a_1^{-2} f + a_7 \\
&t \to -t, \quad x \to x; \quad t \to t, \quad x \to -x; \\
&u \to -u, \quad w \to w; \quad u \to u, \quad w \to -w \\
&t \to t, \quad x \to x/c; \quad u \to w, \quad w \to u, \quad c \to 1/c
\end{aligned}
\tag{2.1}
$$

(a_i ($i = 1, 2, \ldots, 7$) are arbitrary constants satisfying the non-degeneracy condition $a_1 a_4 \neq 0$).

Table 1

$f(u, w)$	Admissible operators
Arbitrary function	$X_1 = \dfrac{\partial}{\partial t}, \quad X_2 = \dfrac{\partial}{\partial x}$
$u^\sigma F(u/w), \quad \sigma \neq 0$ $F(z) \neq \varepsilon\left(\delta - \dfrac{1}{z}\right)^\sigma$ for $\sigma \neq 2$ $F(z) \neq A + \dfrac{B}{z^2} + \dfrac{1}{z}$ for $\sigma = 2$	$X_3 = (\sigma - 2)\left(t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x}\right) - 2\left(u\dfrac{\partial}{\partial u} + w\dfrac{\partial}{\partial w}\right)$
$e^u F(\bar{\delta} u - w)$ $F(z) \neq \varepsilon e^z$	$X_3 = t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} - 2\left(\dfrac{\partial}{\partial u} + \bar{\delta}\dfrac{\partial}{\partial w}\right)$
$F(u/w) + \bar{\varepsilon}\ln u$ $F(z) \neq \bar{\varepsilon}\ln\left(\delta - \dfrac{1}{z}\right)$	$X_3 = t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} + u\dfrac{\partial}{\partial u} + w\dfrac{\partial}{\partial w}$
$F(\delta u - w) + Auw + (\varepsilon - \delta A)u^2/2$ $F'''(z) \neq 0$	$X_i = \varphi_i(t, x)\left(\dfrac{\partial}{\partial u} + \delta\dfrac{\partial}{\partial w}\right), \quad i = 3, 4, 5, 6$
a) $\lambda^2 = \dfrac{\delta\varepsilon - A}{\delta(1 - c^2)} \neq 0, \quad \mu^2 = \dfrac{\delta\varepsilon c^2 - A}{\delta(1 - c^2)} \neq 0$;	$\varphi_3 = \cos\lambda x \cos\mu t, \quad \varphi_4 = \sin\lambda x \sin\mu t$ $\varphi_5 = \cos\lambda x \sin\mu t, \quad \varphi_6 = \sin\lambda x \cos\mu t$
b) $\lambda = 0, \quad \mu^2 = -\dfrac{A}{\delta} = -\varepsilon$;	$\varphi_3 = x\cos\mu t, \quad \varphi_4 = x\sin\mu t$ $\varphi_5 = \cos\mu t, \quad \varphi_6 = \sin\mu t$

Table 1—(*continued*)

$f(u, w)$	Admissible operators
Arbitrary function	$X_1 = \frac{\partial}{\partial t}$, $X_2 = \frac{\partial}{\partial x}$
	$\varphi_5 = \cos \mu t$, $\varphi_6 = \sin \mu t$
c) $\lambda^2 = \frac{A}{\delta c^2} = \varepsilon$, $\mu = 0$	$\varphi_3 = t \cos \lambda x$, $\varphi_4 = t \sin \lambda x$
	$\varphi_5 = \cos \lambda x$, $\varphi_6 = \sin \lambda x$
$F(\delta u - w) + \varepsilon uw - \delta \varepsilon u^2/2$	$X_i = \varphi_i(t,x)\left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w}\right)$, $i = 3,4,5,6$
$F'''(z) \neq 0$	$\varphi_3 = \cos\lambda x \cos\lambda t$, $\varphi_4 = \sin\lambda x \sin\lambda t$,
$\lambda^2 = \frac{\varepsilon}{\delta(c^2 - 1)}$	$\varphi_5 = \cos\lambda x \sin\lambda t$, $\varphi_6 = \sin\lambda x \cos\lambda t$,
$F(\delta u - w) + Au$	$X_i = \varphi_i(t,x)\left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w}\right)$, $i = 3,4,5,6$
$F'''(z) \neq 0$	$\varphi_3 = tx$, $\varphi_4 = t$, $\varphi_5 = x$, $\varphi_6 = 1$
1) $F(z) = \varepsilon z^\sigma + Bz$, $\sigma \neq 0, 1, 2$	$X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{2}{2-\sigma}\left(u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}\right) +$ $+ \frac{1-\sigma}{2-\sigma}\psi(t,x)\left(\frac{\partial}{\partial u} + \delta\frac{\partial}{\partial w}\right)$
2) $F(z) = \varepsilon e^z + Bz$	$X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial w} + \psi(t,x)\left(\frac{\partial}{\partial u} + \delta\frac{\partial}{\partial w}\right)$
3) $F(z) = \varepsilon \ln z + Bz$	$X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w} + \frac{1}{2}\psi(t,x)\left(\frac{\partial}{\partial u} + \delta\frac{\partial}{\partial w}\right)$
4) $F(z) = \varepsilon z \ln z$	$X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2\left(u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}\right) + \frac{\varepsilon}{\delta(c^2-1)} \times$ $\times[(1+\delta^2 c^2)t^2 + (1+\delta^2)x^2]\left(\frac{\partial}{\partial u} + \delta\frac{\partial}{\partial w}\right)$
$\frac{A}{2}u^2 + \frac{B}{2}w^2 + uw + Cu + Dw$	$X_{\varphi,\chi} = \varphi(t,x)\frac{\partial}{\partial u} + \chi(t,x)\frac{\partial}{\partial w}$, where $\varphi_{tt} - \varphi_{xx} = A\varphi + \chi$, $\chi_{tt} - c^2\chi_{xx} = \varphi + B\chi$
a) $AB \neq 1$, $C = D = 0$;	$X_3 = u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}$;
b) $AB = 1$	$X_3 = \left(u + \frac{D - BC}{2(A+B)}t^2\right)\frac{\partial}{\partial u} +$ $+\left(w - \frac{A(D-BC)}{2(A+B)}t^2 + \frac{AC+D}{A+B}\right)\frac{\partial}{\partial w}$

Notes to the table:
1. $c^2 \neq 1, f_{uw}(u, w) \neq 0$.
2. A, B, C, D and σ are arbitrary real constants, δ is a positive and $\tilde{\delta}$ is a non-negative real constant; $\varepsilon = \pm 1$, $\tilde{\varepsilon} = 0, \pm 1$.
3. The constants λ and μ take real or imaginary values.
4. $\psi(t, x) = (1/\delta(c2 - 1))\{[A\delta c^2 + B(1 + \delta^2 c^2)]t^2 + [A\delta + B(1 + \delta^2)]x^2\}$.
5. If $f(u, w) = F(\delta u - w) + Au$, where $F'''(x) \neq 0$ up to equivalence as defined by (2.2), it may be assumed that $A = B = 0$.
6. Additional operators for intersecting subcases are listed only once.

The operator of an admissible point group has the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial w}$$

$$\xi^1 = C_1 t + C_2, \quad \xi^2 = C_1 x + C_3, \quad \eta^1 = C_4 u + \varphi(t,x), \quad \eta^2 = C_4 w + \psi(t,x)$$

where the functions $\varphi(t,x)$ and $\psi(t,x)$ satisfy the classifying equations

$$\varphi_{tt} - \varphi_{xx} = (2C_1 - C_4) f_u + (C_4 u + \varphi) f_{uu} + (C_4 w + \psi) f_{uw}$$

$$\psi_{tt} - c^2 \psi_{xx} = (2C_1 - C_4) f_w + (C_4 u + \varphi) f_{uw} + (C_4 w + \psi) f_{ww}$$

C_i ($i = 1, 2, 3, 4$) are arbitrary constants.

The classifying relation is

$$(b_1 u + b_2) f_u + (b_1 w + b_3) f_w + b_4 f = b_5 u + b_6 w + b_7$$

where b_i ($i = 1, 2, \ldots, 7$) are constant coefficients.

In the subcases the group of equivalence translations may be larger than (2.1). For example, if $f(u, w) = F(\delta u - w) + Au$, where δ and A are constants, $F'''(z) \neq 0$ (the prime stands for differentiation with respect to the argument of the function), all additional continuous transformations have the form

$$\tilde{u} = u + \zeta, \quad \tilde{w} = w + \delta\zeta, \quad \tilde{f} = f + \theta \tag{2.2}$$

where

1. $\theta = 0$, $\zeta = a_8 tx + a_9 t + a_{10} x$ with arbitrary constants a_8, a_9 and a_{10};
2. $\theta = 0$, $\zeta = ((c^2 t^2 + x^2)/(2(c^2 - 1)))(\tilde{A} - A)$, where $\tilde{A}$ satisfies the equation $d\tilde{A}/da = \Phi(\tilde{A}\, c, \delta)$, $\tilde{A}|_{a=0} = A$;
3. $\theta = \kappa(A, c, \delta)(\delta u - w)$, $\zeta = (\kappa(A, c, \delta))/(2\delta(c^2 - 1))[(1 + \delta^2 c^2) t^2 + (1 + \delta^2) x^2]$

where Φ and κ are arbitrary functions of their arguments.

3. Let the force function have the form $f(u, w) = -(\delta u, w)^4$. The variable change

$$\tilde{u} = \delta u, \quad \tilde{w} = w, \quad \tilde{t} = 2t, \quad \tilde{x} = 2x \tag{3.1}$$

then reduces Eqs (1.2) to the form (omitting the tilde)

$$u_{tt} - u_{xx} = -\delta^2 (u - w)^3, \quad w_{tt} - c^2 w_{xx} = (u - w)^3 \tag{3.2}$$

It can be shown that this submodel corresponds to the case in which particles P_n^1 and P_n^2 are coupled together by a Hooke spring, provided that $u_n - w_n| \ll l$, where l is the distance between the chains (the length of the unstretched spring). We must then assume that $\delta^2 = m_2/m_1$ in (3.2).

The following operators constitute a basis for the Lie algebra of Eqs (3.2), taking (3.1) into account

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial u} + \frac{\partial}{\partial w}, \quad X_4 = t\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w}\right), \quad X_5 = x\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w}\right)$$
$$X_6 = tx\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial w}\right), \quad X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - w\frac{\partial}{\partial w} \tag{3.3}$$

Equations (3.2) also admit of reflections

$$t \to -t, \quad x \to x; \quad t \to t, \quad x \to -x; \quad u \to -u, \quad w \to -w \tag{3.4}$$

Let us consider some particular invariant solutions of Eqs (3.2). A solution invariant under the operator $X_2 + \alpha X_5$ ($\alpha = 0$ or 1) has the following form (C_1 and C_2 are arbitrary constants; it may be assumed up to the first equivalence transformation (2.2) that $C_1 = C_2 = 0$)

$$u=\frac{\delta^2}{1+\delta^2}p(t)+\xi,\quad w=-\frac{1}{1+\delta^2}p(t)+\xi$$
$$\xi=\frac{\alpha}{2(1+\delta^2)}[(1+\delta^2c^2)t^2+(1+\delta^2)x^2]+C_1t+C_2 \tag{3.5}$$

where

$$\ddot{p}=-\lambda p^3-\mu,\quad \lambda=1+\delta^2,\quad \mu=\alpha(c^2-1) \tag{3.6}$$

The energy integral of Eq. (3.6) can be written as

$$(\dot{p})^2+\frac{\lambda}{2}p^4+2\mu p-2E^*=2E,\quad E^*=-\frac{3}{4}\mu\left(\frac{\mu}{\lambda}\right)^{1/3}$$

To any number $E>0$ there corresponds a bounded periodic solution of Eq. (3.6) which describes non-linear oscillations about the equilibrium position $p=-(\mu/\lambda)^{1/3}$ in the domain $p\in[p_1,p_2]$, where p_1 and p_2 are the least and greatest real roots, respectively, of the equation $\lambda p^4/4+\mu p-E^*=E$.

If $\alpha=0$, the solution (3.5) is spatially homogeneous (this may be regarded as the case of extremely long waves) and may be expressed in terms of Jacobi elliptic functions, for example, as

$$u=\frac{\delta^2}{1+\delta^2}\left(\frac{4E}{1+\delta^2}\right)^{1/4}\mathrm{cn}[(4E(1+\delta^2))^{1/4}t,k]=-\delta^2w,\quad k^2=\frac{1}{2}$$

The particles in the chains move in opposite directions, the oscillation amplitudes being the same only in the case $\delta^2=1$ $(m_1=m_2)$. In the limit as $\delta\to 0$ $(m_1\gg m_2)$, we obtain a solution describing oscillations of the "lower" particles only

$$u=0,\quad w=(4E)^{1/4}\mathrm{cn}[(4E)^{1/4}t,k],\quad k^2=1/2$$

A solution invariant under the operator $X_1+\upsilon X_2+\alpha X_4$ ($\alpha=0$ or 1, where υ is an arbitrary constant) has the following form when $\upsilon^2\neq 1, c^2, (1+\delta^2c^2)/(1+\delta^2)$ (up to the first equivalence transformation (2.2), it may be assumed that $C_1=C_2=0$)

$$u=\sigma^{-1}\{\delta^2(\upsilon^2-c^2)p(x-\upsilon t)+\eta\},\quad w=\sigma^{-1}\{(1-\upsilon^2)p(x-\upsilon t)+\eta\} \tag{3.7}$$
$$\sigma=\delta^2(\upsilon^2-c^2)+\upsilon^2-1$$
$$\eta=\alpha(1+\delta^2)\upsilon tx-\frac{\alpha}{2}[(1+\delta^2)t^2+(1+\delta^2)x^2]+C_1x-\upsilon t)+C_2$$

where

$$p''=-\lambda_1p^3-\mu_1,\quad \lambda_1=\frac{\sigma}{(\upsilon^2-1)(\upsilon^2-c^2)},\quad \mu_1=\frac{\alpha(1-c^2)}{(\upsilon^2-1)(\upsilon^2-c^2)} \tag{3.8}$$

If $\lambda_1>0$, formula (3.8) is identical with (3.6), apart from the notation. Consequently, the solutions of Eq. (3.8) in this case are bounded and describe periodic travelling waves. The waves may propagate at velocities in the range

$$\upsilon^2\in\,]S,M[\,\cup\,]L,+\infty[\tag{3.9}$$
$$\left(S=\min\{1,c^2\},\quad M=\frac{1+\delta^2c^2}{1+\delta^2},\quad L=\max\{1,c^2\}\right)$$

In particular, when $\alpha=0$ we obtain a solution in the form of conoidal waves

$$u = U\left(\frac{4E}{\lambda_1}\right)^{1/4} \mathrm{cn}[(4E\lambda_1)^{1/4}(x - \upsilon t), k] = Ww$$

$$U = \frac{\delta^2(\upsilon^2 - c^2)}{\sigma}, \quad W = \frac{\delta^2(\upsilon^2 - c^2)}{1 - \upsilon^2}, \quad k^2 \frac{1}{2}, \quad E > 0$$

If $\lambda_1 < 0$, then (3.8) may be regarded as the equation of motion of a particle in the field of a potential well $U(p) = -|\lambda_1| p^4/4 + \mu_1 p$, in which case no non-trivial bounded solutions exist.

Using the reflections (3.4), which are admissible for equations (3.2), one can obtain solutions with other sign combinations.

4. Let the force function have the form $f(u, w) = \cos(\delta u - w) - 1$. The variable change $\tilde{u} = \delta u$, $\tilde{w} = w$ will then reduce Eqs (1.2) to the following form (omitting the tilde)

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \quad w_{tt} - c^2 w_{xx} = \sin(u - w) \tag{4.1}$$

This submodel is a possible generalization of the Frenkel–Kontorova (FK) dislocation model [4, 5]. While the FK model is concerned with the displacement of one part of a crystal relative to another part, which is treated as if motionless, here it is assumed that both parts of the crystal are deformed (Fig. 2). Assuming that the half-planes $x = \mathrm{const}, y > 0$ and $x = \mathrm{const}, y < 0$ are moving linearly along X, we obtain a model of coupled particle chains (AB and CD in Fig. 2). Assuming that the potential of the non-linear interaction in long-wave motions is a harmonic function of the particle displacements in the chains and changing to dimensionless variables, one obtains Eqs (4.1) with $\delta^2 = m_2/m_1$. The dimensional and dimensionless variables are related by

$$\tilde{t} = \chi t, \quad \tilde{x} = \chi c_1 x, \quad \tilde{u} = \frac{a}{2\pi} u, \quad \tilde{w} = \frac{a}{2\pi} w, \quad \chi = \left(\frac{a m_2}{2\pi\tau}\right)^{1/2}$$

where τ is the interaction constant, which has the dimensions of force.

The transition from (4.1) to the FK model is obtained by letting $\delta \to 0$ ($m_1 \gg m_2$). Indeed, setting $u = 0$, we obtain the well-known Sine–Gordon equation for the displacements of the particles of mass m_2 in this case, as obtained in the FK model.

The first six operators of (3.3) constitute a basis of the Lie algebra of Eqs (4.1). Equations (4.1) also admit of reflections (3.4).

Let us consider a few solutions of Eqs (4.1). A solution invariant under the operator $X_2 + \alpha X_5$ (where α is a non-negative constant) has the form (3.5), where $p(t)$ satisfies the equation

$$\ddot{p} = -\lambda \sin p - \mu, \quad \lambda = 1 + \delta^2, \quad \mu = \alpha(c^2 - 1) \tag{4.2}$$

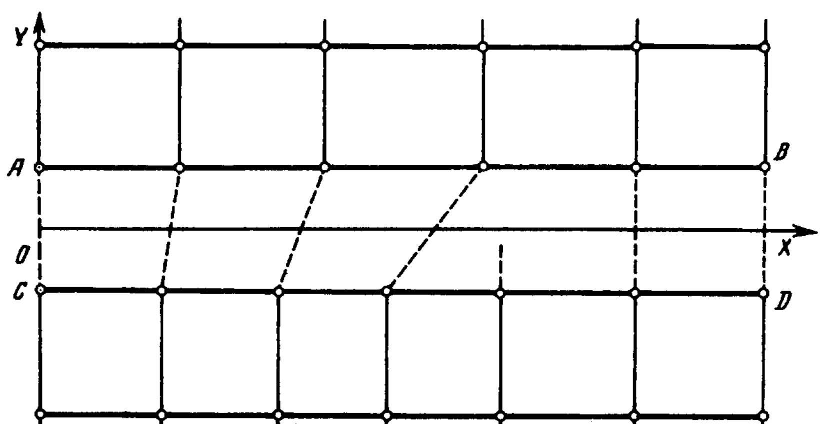

Fig. 2.

The energy integral of Eq. (4.2) may be written as

$$(\dot{p})^2 + 2\lambda(1-\cos p) + 2\mu p - 2E^* = 2E \tag{4.3}$$

$$E^* = \lambda - (\lambda^2 - \mu^2)^{1/2} + \mu p_0, \quad p_0 = -\arcsin \mu / \lambda$$

If

$$|\mu| < \lambda \tag{4.4}$$

then to any number E in the domain

$$0 < E < 2\lambda - \pi|\mu| - 2R^* \tag{4.5}$$

there corresponds a bounded periodic solution of Eq. (4.2) describing non-linear oscillations about the equilibrium position $p = p_0$ in the domain $p \in [p_1, p_2]$, where p_1 and p_2 are the least and greatest real roots, respectively, of the equation $\lambda(1 - \cos p) + \mu p - E^* = E$ in the interval $]-\pi + \arcsin a/\lambda, \pi + \arcsin a/\lambda[$. It is assumed that $p_0 \in]-\pi/2, \pi/2[$.

Let $\alpha = 0$. Then Eqs (4.2) describe the oscillations of a mathematical pendulum and can be integrated in elliptic functions (see, for example, [6]). The corresponding spatially homogeneous solution of Eqs (4.1), which describes oscillations of the particles about their equilibrium positions, is

$$u = \frac{2\delta^2}{1+\delta^2} \arcsin\{k \,\mathrm{sn}[(1+\delta^2)^{1/2} t, k]\} = -\delta^2 w, \quad 0 < k < 1$$

A solution invariant under the operator $X_1 + \upsilon X_2 + \alpha X_4$ (where υ is an arbitrary constant and α is a non-negative constant) has the form (3.7) when $\upsilon^2 \neq 1, c^2, (1 + \delta^2 c^2)/(1 + \delta^2)$, where the function $p(x - \upsilon t)$ satisfies the equation

$$p'' = -\lambda_1 \sin p - \mu_1 \tag{4.6}$$

where λ_1 and μ_1 are the same as in (3.8).

If $\lambda_1 < 0$, the change of variable $\tilde{p} = p + \pi$ reduces Eq. (4.6) to the form $\tilde{p}'' = -|\lambda_1| \sin \tilde{p} - \mu_1$. Therefore, we may assume when analysing Eq. (4.6) that $\lambda_1 > 0$, and the equation is then identical, apart from notation, with (4.3). Consequently, when conditions of the same type as (4.4) and (4.5) are satisfied, the solutions of Eq. (4.6) are bounded and describe periodic travelling waves. When $\lambda_1 > 0$ the waves propagate at velocities in the range (3.9); if $\lambda_1 < 0$, the velocities are in the range

$$\upsilon^2 \in [0, S[\cup]M, L[\tag{4.7}$$

In particular, when $\alpha = 0$ we obtain the following solutions of Eqs (4.1) in the form of periodic travelling waves.

"Fast" waves propagating at velocities in the range (3.9)

$$u = 2U \arcsin\{k \,\mathrm{sn}[\lambda_1^{1/2}(x - \upsilon t), k]\} = Ww$$

"Slow" waves propagating at velocities in the range (4.7)

$$u = 2U \arcsin\{\mathrm{dn}[|\lambda_1|^{1/2}(x - \upsilon t), k]\} = Ww$$

where $0 < k < 1$.

In the limiting non-linear case $k = 1$ the periodic "slow" waves become solitary travelling waves (solitons)

$$u = 4U \,\mathrm{arctg}\{\exp[|\lambda_1|^{1/2}(x - \upsilon t)]\} = Ww$$

Using the reflections (3.4), one can obtain solutions with other sign combinations.

If $m_1 \gg m_2$ ($\delta \to 0$), the solitons may propagate at velocities $\upsilon^2 \in [0, c^2[$. In that case the displacement of particles in the crystal is independent of the velocity of wave propagation. If the masses m_1 and m_2 are commensurable, the solitons may propagate at velocities in the range (4.7), i.e. we have

$$\upsilon^2 \in [0, 1[\cup]M, c^2[\quad \text{if} \quad c^2 > 1; \quad \upsilon^2 \in [0, c^2[\cup]M, 1[\quad \text{if} \quad c^2 < 1$$

Consequently, if the acoustic velocities of non-interacting chains are different ($c^2 \neq 1$), a gap will appear in the velocity spectrum of the solitons, i.e. the system will act as a kind of soliton filter. Here the relative displacement ("upper" particles relative to "lower") will remain the same as in the FK model (per period of the chain), but the absolute displacement will depend on the velocity of wave propagation.

5. On the basis of the solutions constructed in Sections 3 and 4 and the equivalence transformations just found, one can obtain certain solutions describing the long-wave dynamics of the above mechanical system when there are additional shear forces.

Using (3.5) and the third equivalence transformation (2.2), with variables changed by the rules

$$u \to u/\delta, \quad \tilde{u} \to \tilde{u}/\delta \tag{5.1}$$

and, $\kappa = \alpha(1 - c^2)/(1 + \delta^2)$, we obtain a solution

$$u = \frac{\delta^2}{1 + \delta^2} p(t) = -\delta^2 w$$

[the function $p(t)$ satisfies Eq. (3.6) or Eq. (4.2)] which describes oscillations of the particles in the "upper" and "lower" chains, on the assumption that both chains are subject to additional shear forces of equal magnitude but in opposite directions.

Using (3.5) and the second equivalence transformation (2.2), with variables changed as in (5.1) and

$$A = 0, \quad \tilde{A} = \frac{\alpha(1 - c^2)}{\delta c^2} M$$

we obtain a solution

$$u = \frac{\delta^2}{1+\delta^2} p(t) + \omega, \quad w = -\frac{1}{1+\delta^2} p(t) + \omega, \quad \omega = \frac{\alpha(c^2 - 1)}{2c^2(1+\delta^2)} x^2$$

($p(t)$ satisfies Eq. (3.6) or Eq. (4.2)) which describes the displacements of the particles when only the "upper" chain is subject to additional shear forces.

Analogous solutions may be constructed for travelling waves. Indeed, using (3.7) and the first and third equivalence transformations, with the change of variable (5.1) and the substitution

$$a_8 = \frac{\alpha(1+\delta^2)\upsilon}{\delta\sigma}, \quad a_9 = a_{10} = 0, \quad \kappa = \frac{\alpha(c^2 - 1)}{\sigma},$$

we obtain a solution

$$u = Up(x - \upsilon t) = Ww$$

(the function $p(x - \upsilon t)$ satisfies Eq. (3.8) or Eq. (4.2)) which describes travelling waves in the chains, when the latter are subject to shear forces of equal magnitude but in opposite directions.

Using the first and second transformations (2.2) with the same values of a_8, a_9 and a_{10} and with

$$A = 0, \quad \tilde{A} = \frac{\alpha(c^2 - 1)(1+\delta^2 c^2)}{\delta c^2 \sigma}$$

and making the change (5.1), we obtain from (3.7) a solution

$$u = \sigma^{-1}\{\delta^2(\upsilon^2 - c^2)p(x - \upsilon t) - (1 + \delta^2)\omega\}, \quad w = \sigma^{-1}\{(1 - \upsilon^2)p(x - \upsilon t) - (1 + \delta^2)\omega\}$$

($p(x - \upsilon t)$ satisfies Eq. (3.8) or Eq. (4.6)). Here only the "upper" chain is subject to additional shear forces.

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